

COHOMOLOGY STRUCTURES OF A POISSON ALGEBRA: II

YAN-HONG BAO AND YU YE

ABSTRACT. We introduce for any Poisson algebra a bicomplex of free Poisson modules, and use it to show that the Poisson cohomology theory introduced in the paper "[M. Flato, M. Gerstenhaber and A. A. Voronov, Cohomology and Deformation of Leibniz Pairs, *Lett. Math. Phys.* **34** (1995) 77–90]" is given by certain derived functor. Moreover, by constructing a long exact sequence connecting Poisson cohomology groups and Yoneda-extension groups of certain quasi-Poisson modules, we provide a way to compute this Poisson cohomology via the Lie algebra cohomology and the Hochschild cohomology.

1. INTRODUCTION

M. Flato, M. Gerstenhaber and A.A.Voronov developed a cohomology theory and a formal deformation theory for Poisson algebras [3]. They showed that this cohomology controls those formal deformations such that the associative multiplication and the Lie bracket are simultaneously deformed. We call this cohomology FGV-Poisson cohomology, or simply Poisson cohomology. Note that when restricted to commutative Poisson algebras, the FGV-Poisson cohomology is different from the usual Lichnerowicz-Poisson cohomology, see Section 5 for an explanation.

We would like to mention that Kontsevich's deformation quantization can be understood as a special case of formal deformations of Poisson algebras, see Section 4 below for more detail. Moreover, a necessary condition for the existence of deformation quantization is given there by using the Poisson cohomology groups.

Since the concept of a module of a Poisson algebra is meaningful, it was also supposed by Flato-Gerstenhaber-Voronov to use Yoneda-extensions or derived functors to define certain cohomology theory for Poisson algebras. An immediate question is whether it is the same as the FGV-Poisson cohomology, or in other words, whether the FGV-Poisson cohomology is exactly given by usual Yoneda-extensions or derived functors. In this note, we will give an affirmative answer to this question, and provide a way to compute the FGV-Poisson cohomology via Hochschild cohomology and Chevalley-Eilenberg cohomology.

Throughout \mathbb{K} will be a field of characteristic 0, and all unadorned Hom and \otimes will be $\text{Hom}_{\mathbb{K}}$ and $\otimes_{\mathbb{K}}$ respectively. Let A be a Poisson algebra over \mathbb{K} . Following [10], one can define its Poisson enveloping algebra \mathcal{P} in a natural way. By definition \mathcal{P} is an associative algebra such that the category of left \mathcal{P} -modules is isomorphic to the category of Poisson A -modules. We need also the concept of quasi-Poisson modules as introduced in the same work [10]. It was shown there the category of

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E-mail: yhbao@ustc.edu.cn (Y.-H. Bao), yeyu@ustc.edu.cn (Y.Ye).

quasi-Poisson A -modules is isomorphic to the module category of an associative algebra \mathcal{Q} , which is called the quasi-Poisson enveloping algebra of A .

In our previous work [1], we have introduced a bicomplex of free quasi-Poisson A -modules, whose total complex gives a free resolution of A as a quasi-Poisson module and hence calculates the quasi-Poisson cohomology. Inspired by this construction, we obtain a bicomplex $C_{\bullet,\bullet}$ of free Poisson A -modules from the foregoing one by applying the functor $\mathcal{P} \otimes_{\mathcal{Q}} -$, and by erasing one row of the bicomplex. Recall that \mathcal{P} is a quotient algebra of \mathcal{Q} and can be viewed as a \mathcal{Q} -module. The reason of the elision of one row is that concerning the deformation theory, the Lie bracket and the Lie module structure of A on itself can not be deformed independently.

For any Poisson module M , we show that the bicomplex $\mathrm{Hom}_{\mathcal{P}}(C_{\bullet,\bullet}, M)$ coincides with the bicomplex $\tilde{C}^{\bullet,\bullet}(A; M)$ as given in [3, Section 5]. Thus the FGV-Poisson cohomology is exactly defined by the derived functor $\mathrm{RHom}_{\mathcal{P}}(\chi_{\bullet}(A), -)$, where $\chi_{\bullet}(A) = \mathrm{Tot}(C_{\bullet,\bullet})$ is the total complex of the double complex $C_{\bullet,\bullet}$. In this sense, we call $\chi_{\bullet}(A)$ the *characteristic complex* of the Poisson algebra A .

We mention that the characteristic complex is not quasi-isomorphic to a Poisson module in general. However, there is a long exact sequence of abelian groups with terms involving homology groups of the characteristic complex and certain torsion groups in the category of quasi-Poisson modules.

A key observation is that FGV-Poisson cohomology can be calculated via quasi-Poisson cohomologies, although it is defined only with coefficients in Poisson modules. In fact, we have a long exact sequence connecting the Poisson cohomology groups and the extension groups of certain quasi-Poisson modules. This could be very useful, the reason is that a Poisson enveloping algebra is much more difficult to handle, we even do not have a nice basis for it in general.

The quasi-Poisson case is easier. Recall that A^e is a $\mathcal{U}(A)$ -module algebra, and by construction $\mathcal{Q} = A^e \# \mathcal{U}(A)$ is the smash product algebra. Recall that in [1], we have obtained a Grothendieck spectral sequence, calculating $\mathrm{Ext}_{\mathcal{Q}}^*(-, -)$ via $\mathrm{Ext}_{A^e}^*(-, -)$ and the Lie algebra cohomology $\mathrm{Ext}_{\mathcal{U}(A)}^*(\mathbb{K}, -)$.

In summary, we may reduce the calculation of FGV-Poisson cohomology to the cohomology of the quasi-Poisson enveloping algebra \mathcal{Q} , and hence to the Hochschild cohomology of A and the Chevalley-Eilenberg cohomology of the Lie algebra A , on which we have better knowledge.

The paper is organized as follows. In section 2, we recall some basic definitions of Poisson algebras, Poisson modules and Poisson enveloping algebras. Section 3 deals with the construction of a bicomplex of free Poisson modules for a Poisson algebra A , whose total complex computes the FGV-Poisson cohomologies.

In section 4, we will briefly recall the definition of formal deformation of Poisson algebras, and show how to understand Kontsevich's deformation quantization as a special case of formal deformation of Poisson algebras in our sense. Moreover, we will explain in more detail how the FGV-Poisson cohomology groups control the formal deformations of Poisson algebras. We also compare the FGV-Poisson cohomology groups with the Lichnerowicz-Poisson cohomology groups for commutative Poisson algebras in Section 5.

In Section 6 we show a long exact sequence related to homology groups of the characteristic complex. We also obtain a long exact sequence with terms involving the FGV-Poisson cohomology groups and Yoneda-extension groups of certain

quasi-Poisson modules, and apply it to calculate the Poisson cohomology groups. Examples are also provided there.

In the last section, we study the the standard Poisson algebra of the matrix algebra $M_2(\mathbb{K})$. We calculate its Poisson cohomology groups of lower degrees. Moreover, we show that in this special case, any Poisson 2-cocycle lifts to a formal deformation.

2. PRELIMINARIES

In this paper, we assume that all associative algebras have a multiplicative identity element.

A triple $(A, \cdot, \{-, -\})$ is called a *Poisson algebra* over \mathbb{K} , if (A, \cdot) is an associative \mathbb{K} -algebra (not necessarily commutative), $(A, \{-, -\})$ is a Lie algebra over \mathbb{K} , and the Leibniz rule $\{ab, c\} = a\{b, c\} + \{a, c\}b$ holds for all $a, b, c \in A$. A *quasi-Poisson A -module* M is both an A - A -bimodule and a Lie module over $(A, \{-, -\})$ with the action given by $\{-, -\}_* : A \times M \rightarrow M$, which satisfies

$$\begin{aligned} \{a, bm\}_* &= \{a, b\}m + b\{a, m\}_*, \\ \{a, mb\}_* &= m\{a, b\} + \{a, m\}_*b \end{aligned}$$

for all $a, b \in A$ and $m \in M$. If moreover,

$$\{ab, m\}_* = a\{b, m\}_* + \{a, m\}_*b$$

holds for all $a, b \in A$ and $m \in M$, then M is called a *Poisson A -module*. Let M, N be (quasi-)Poisson modules. A homomorphism of (quasi-)Poisson A -modules is a \mathbb{K} -linear function $f : M \rightarrow N$ which is a homomorphism of both A - A -bimodules and Lie modules.

The following convention is handy in calculation, and we refer to [10] for more details.

Denote by A^{op} the opposite algebra of the associative algebra A . To avoid confusion, we usually use a to denote an element in A and a' its counterpart in A^{op} . The algebra $A^e = A \otimes A^{\text{op}}$ is called the *enveloping algebra* of the associative algebra A . Denote by $\mathcal{U}(A)$ the *universal enveloping algebra* of the Lie algebra $(A, \{-, -\})$. It is well-known that the category of Lie modules over A is isomorphic to the category of left $\mathcal{U}(A)$ -modules.

Let $\{v_i \mid i \in S\}$ be a \mathbb{K} -basis of A indexed by an ordered set S with a total ordering \leq . For any r -tuple $\alpha = (i(1), \dots, i(r)) \in S^r$, the corresponding element of $v_{i(1)} \otimes \dots \otimes v_{i(r)}$ in $\mathcal{U}(A)$ is denoted by $\vec{\alpha}$, and r is called the degree of α . The empty sequence is denoted by \emptyset and we write $\mathbb{1} = 1_{\mathcal{U}(A)} = \vec{\emptyset}$ for brevity. Then $\{\vec{\alpha} \mid i_1 \leq \dots \leq i_r, r \geq 0\} \cup \{\mathbb{1}\}$ form a PBW-basis of $\mathcal{U}(A)$.

$\mathcal{U}(A)$ is known to be a cocommutative Hopf algebra. The comultiplication is given by $\Delta(\vec{\alpha}) = \sum_{\alpha = \alpha_1 \sqcup \alpha_2} \vec{\alpha}_1 \otimes \vec{\alpha}_2$, where the sum is taken over all possible ordered bipartition. The Lie bracket makes A a Lie module over A , or equivalently, a $\mathcal{U}(A)$ -module. Hence the usual tensor product makes A^e a Lie module. Moreover, the cocommutativity of $\mathcal{U}(A)$ makes A^e a $\mathcal{U}(A)$ -module algebra. For a definition of ordered partition, we refer to [10].

Definition 2.1. ([10]) Let $A = (A, \cdot, \{-, -\})$ be a Poisson algebra. The smash product $A^e \# \mathcal{U}(A)$ is called the *quasi-Poisson enveloping algebra* of A and denoted by \mathcal{Q} . The *Poisson enveloping algebra* of A , denoted by \mathcal{P} , is defined to be the quotient algebra \mathcal{Q}/J , where J is the ideal of \mathcal{Q} generated by $\{1_A \otimes 1'_A \# (a \cdot b) - a \otimes 1'_A \# b - 1_A \otimes b' \# a \mid a, b \in A\}$.

Theorem 2.2. ([10]) *The category of quasi-Poisson modules over A is isomorphic to the category of left \mathcal{Q} -modules, and the category of Poisson modules over A is isomorphic to the category of left \mathcal{P} -modules.*

Given a quasi-Poisson A -module M , one defines a \mathcal{Q} -module M by setting

$$(a \otimes b' \# \vec{\alpha})m = a \vec{\alpha}(m)b$$

for all $m \in M$ and $a \otimes b' \# \vec{\alpha} \in \mathcal{Q}$. Conversely, given a left \mathcal{Q} -module M , we set

$$am = (a \otimes 1'_A \# 1)m, \quad ma = (1_A \otimes a' \# 1)m, \quad \{a, m\}_* = (1_A \otimes 1'_A \# a)m$$

for all $m \in M, a \in A$ to obtain a quasi-Poisson module over A . The correspondence of Poisson modules and \mathcal{P} -modules is given similarly. For simplicity, we write $a \otimes b' \# \vec{\alpha} + J$ as $a \otimes b' \# \vec{\alpha}$ when no confusion can arise.

The following bicomplex is given by taking tensor of the bar resolution of the A^e -module A and the Koszul resolution of the $\mathcal{U}(A)$ -module \mathbb{K} , where we write $A^i = \otimes^i A$ and $\wedge^j = \wedge^j A$ for brevity.

$$\begin{array}{ccccccc}
 & \dots & & \dots & & \dots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longleftarrow & A^4 \otimes \mathcal{U}(A) & \xleftarrow{\eta_{2,0}^H} & A^4 \otimes \mathcal{U}(A) \otimes \wedge^1 & \xleftarrow{\eta_{2,1}^H} & A^4 \otimes \mathcal{U}(A) \otimes \wedge^2 \longleftarrow \dots \\
 & & \eta_{1,0}^V \downarrow & & \eta_{1,1}^V \downarrow & & \eta_{1,2}^V \downarrow \\
 (2.1) \quad 0 & \longleftarrow & A^3 \otimes \mathcal{U}(A) & \xleftarrow{\eta_{1,0}^H} & A^3 \otimes \mathcal{U}(A) \otimes \wedge^1 & \xleftarrow{\eta_{1,1}^H} & A^3 \otimes \mathcal{U}(A) \otimes \wedge^2 \longleftarrow \dots \\
 & & \eta_{0,0}^V \downarrow & & \eta_{0,1}^V \downarrow & & \eta_{0,2}^V \downarrow \\
 0 & \longleftarrow & A^2 \otimes \mathcal{U}(A) & \xleftarrow{\eta_{0,0}^H} & A^2 \otimes \mathcal{U}(A) \otimes \wedge^1 & \xleftarrow{\eta_{0,1}^H} & A^2 \otimes \mathcal{U}(A) \otimes \wedge^2 \longleftarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Its total complex gives a free resolution of A as a \mathcal{Q} -module. For any quasi-Poisson module M , after applying the functor $\text{Hom}_{\mathcal{Q}}(-, M)$ to the total complex, we obtain the quasi-Poisson complex with coefficients in M , whose cohomology group is isomorphic the extension group $\text{Ext}_{\mathcal{Q}}^*(A, M)$, and called the quasi-Poisson cohomology group with coefficients in M , see [1, Section 2].

3. THE POISSON COHOMOLOGY

3.1. Characteristic complexes of Poisson algebras. Let $(A, \cdot, \{-, -\})$ be a Poisson algebra and \mathcal{P} the Poisson enveloping algebra of A . Set

$$C_{i,j} = \begin{cases} \mathcal{P} \otimes \wedge^j, & i = 0, j \geq 0, \\ \mathcal{P} \otimes A^{i+1} \otimes \wedge^{j-1}, & i \geq 1, j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the following diagram

$$\begin{array}{ccccccc}
 & \dots & & \dots & & \dots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longleftarrow & \mathcal{P} \otimes A^3 & \xleftarrow{\delta_{2,1}^H} & \mathcal{P} \otimes A^3 \otimes \wedge^1 & \xleftarrow{\delta_{2,2}^H} & \mathcal{P} \otimes A^3 \otimes \wedge^2 \longleftarrow \dots \\
 & & \delta_{1,1}^V \downarrow & & \delta_{1,2}^V \downarrow & & \delta_{1,3}^V \downarrow \\
 0 & \longleftarrow & \mathcal{P} \otimes A^2 & \xleftarrow{\delta_{1,1}^H} & \mathcal{P} \otimes A^2 \otimes \wedge^1 & \xleftarrow{\delta_{1,2}^H} & \mathcal{P} \otimes A^2 \otimes \wedge^2 \longleftarrow \dots \\
 & & \delta_{0,1}^V \downarrow & & \delta_{0,2}^V \downarrow & & \delta_{0,3}^V \downarrow \\
 \mathcal{P} & \xleftarrow{\delta_{0,0}^H} & \mathcal{P} \otimes \wedge^1 & \xleftarrow{\delta_{0,1}^H} & \mathcal{P} \otimes \wedge^2 & \xleftarrow{\delta_{0,2}^H} & \mathcal{P} \otimes \wedge^3 \longleftarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where $\delta_{i,j}^V : C_{i+1,j} \rightarrow C_{i,j}$ and $\delta_{i,j}^H : C_{i,j+1} \rightarrow C_{i,j}$ are \mathcal{P} -homomorphisms given by

$$\begin{aligned}
 & \delta_{i,j}^V((1_A \otimes 1'_A \# \mathbb{1}) \otimes a_0 \otimes \dots \otimes a_i \otimes \omega^j) \\
 &= (-1)^i [(a_0 \otimes 1'_A \# \mathbb{1}) \otimes a_1 \otimes \dots \otimes a_i \otimes \omega^j \\
 & \quad + \sum_{k=0}^{i-1} (-1)^{k+1} (1_A \otimes 1'_A \# \mathbb{1}) \otimes a_0 \otimes \dots \otimes a_k a_{k+1} \otimes \dots \otimes a_i \otimes \omega^j \\
 & \quad + (-1)^{i+1} (1 \otimes a'_i \# \mathbb{1}) \otimes a_0 \otimes \dots \otimes a_{i-1} \otimes \omega^j]
 \end{aligned}$$

for $i \geq 2$ and $j \geq 0$,

$$\begin{aligned}
 & \delta_{1,j}^V((1_A \otimes 1'_A \# \mathbb{1}) \otimes a_0 \otimes a_1 \otimes \omega^j) \\
 &= (a_0 \otimes 1'_A \# \mathbb{1}) \otimes (a_1 \wedge \omega^j) - (1_A \otimes 1'_A \# \mathbb{1}) \otimes (a_0 a_1 \wedge \omega^j) + (1_A \otimes a'_1 \# \mathbb{1}) \otimes (a_0 \wedge \omega^j)
 \end{aligned}$$

for $i = 0$, $j \geq 1$ and $\omega^j \in \wedge^j$, and

$$\begin{aligned}
 & \delta_{i,j}^H((1_A \otimes 1'_A \# \mathbb{1}) \otimes \theta^i \otimes (x_0 \wedge \dots \wedge x_j)) \\
 &= \sum_{k=0}^j (-1)^k (1_A \otimes 1'_A \# x_k) \otimes \theta^i \otimes (x_0 \wedge \dots \widehat{x_k} \dots \wedge x_j) \\
 & \quad + \sum_{0 \leq p < q \leq j} (1_A \otimes 1'_A \# \mathbb{1}) \otimes \theta^i \otimes (\{x_p, x_q\} \wedge x_0 \wedge \dots \widehat{x_p} \dots \widehat{x_q} \dots \wedge x_j)
 \end{aligned}$$

for any $i \geq 0$, $j \geq 1$ and $\theta^i \in A^i$.

Proposition-Definition 3.1. $C_{\bullet,\bullet} = (C_{i,j}, \delta_{i,j}^H, \delta_{i,j}^V)$ is a bicomplex of free \mathcal{P} -modules. Its total complex is called the characteristic complex of the Poisson algebra A and denoted by $\chi_{\bullet}(A)$.

Proof. Notice that \mathcal{P} is a quotient algebra of \mathcal{Q} and hence is viewed as a \mathcal{Q} -module. By applying the functor $\mathcal{P} \otimes_{\mathcal{Q}} -$ the bicomplex (2.1) and the natural isomorphism of left \mathcal{P} -modules $\mathcal{P} \otimes_{\mathcal{Q}} \mathcal{Q} \otimes (A^i \otimes \wedge^j) \cong \mathcal{P} \otimes A^i \otimes \wedge^j$, we obtain the following bicomplex

of Poisson modules

$$(3.1) \quad \begin{array}{ccccccc} & \cdots & & \cdots & & \cdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \leftarrow & \mathcal{P} \otimes A^3 & \xleftarrow{\bar{\eta}_{3,0}^H} & \mathcal{P} \otimes A^3 \otimes \wedge^1 & \xleftarrow{\bar{\eta}_{3,1}^H} & \mathcal{P} \otimes A^3 \otimes \wedge^2 & \leftarrow \cdots \\ & \downarrow \bar{\eta}_{2,0}^V & & \downarrow \bar{\eta}_{2,1}^V & & \downarrow \bar{\eta}_{2,2}^V & \\ 0 \leftarrow & \mathcal{P} \otimes A^2 & \xleftarrow{\bar{\eta}_{2,0}^H} & \mathcal{P} \otimes A^2 \otimes \wedge^1 & \xleftarrow{\bar{\eta}_{2,1}^H} & \mathcal{P} \otimes A^2 \otimes \wedge^2 & \leftarrow \cdots \\ & \downarrow \bar{\eta}_{1,0}^V & \searrow \delta_{0,1}^V & \downarrow \bar{\eta}_{1,1}^V & \searrow \delta_{0,2}^V & \downarrow \bar{\eta}_{1,2}^V & \searrow \delta_{0,3}^V \\ 0 \leftarrow & \mathcal{P} \otimes A & \xleftarrow{\varphi_0} & \mathcal{P} \otimes A \otimes \wedge^1 & \xleftarrow{\varphi_1} & \mathcal{P} \otimes A \otimes \wedge^2 & \leftarrow \cdots \\ & \downarrow & \searrow \varphi_0 & \downarrow & \searrow \varphi_1 & \downarrow & \searrow \varphi_2 \\ 0 \leftarrow & \mathcal{P} & \xleftarrow{\bar{\eta}_{0,0}^H} & \mathcal{P} \otimes \wedge^1 & \xleftarrow{\bar{\eta}_{0,1}^H} & \mathcal{P} \otimes \wedge^2 & \leftarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

where $\bar{\eta}_{i,j}^V, \bar{\eta}_{i,j}^H$ are induced by $\text{id}_{\mathcal{P}} \otimes \eta_{i,j}^V$ and $\text{id}_{\mathcal{P}} \otimes \eta_{i,j}^H$ respectively. Erasing all of dashed arrows, we obtain the diagram $C_{\bullet,\bullet}$, and $\delta_{0,j}^V = \varphi_{j-1} \bar{\eta}_{1,j-1}^V$, $\delta_{i,j}^V = \bar{\eta}_{i+1,j-1}^V$ ($i \geq 2$), and $\delta_{0,j}^H = \bar{\eta}_{0,j}$, $\delta_{i,j}^H = \bar{\eta}_{i+1,j-1}^H$ ($i \geq 1$), where

$$\varphi_j(1_A \otimes 1'_A \# 1 \otimes a \otimes \omega^j) = 1_A \otimes 1'_A \# 1 \otimes (a \wedge \omega^j).$$

It is easily seen that $\delta_{i,j}^H \delta_{i,j+1}^H = 0$, $\delta_{i,j}^V \delta_{i+1,j}^V = 0$, $\delta_{i,j-1}^V \delta_{i,j}^H + \delta_{i-1,j}^H \delta_{i,j}^V = 0$ for all $i \geq 2$, and $\delta_{0,j}^V \delta_{1,j}^V = \varphi_{j-1} \bar{\eta}_{1,j-1}^V \bar{\eta}_{2,j-1}^V = 0$. It remains to check $\delta_{0,1}^V \delta_{1,j}^H + \delta_{0,j}^H \delta_{0,j+1}^V$ for all $j \geq 1$. We only show $j = 1$ case, and the proof for general cases is exactly the same and left to the readers. By definition we have

$$\begin{aligned} & (\delta_{0,1}^V \delta_{1,1}^H)(1_A \otimes 1'_A \# 1 \otimes a \otimes b \otimes x) \\ &= \delta_{0,1}^V(1_A \otimes 1'_A \# x \otimes a \otimes b) \\ &= \delta_{0,1}^V[(1_A \otimes 1'_A \# x)(1_A \otimes 1'_A \# 1 \otimes a \otimes b) \\ &\quad - 1_A \otimes 1'_A \# 1 \otimes \{x, a\} \otimes b - 1_A \otimes 1'_A \# 1 \otimes a \otimes \{x, b\}] \\ &= (1_A \otimes 1'_A \# x)[a \otimes 1_A \# 1 \otimes b - 1_A \otimes 1'_A \# 1 \otimes ab + 1_A \otimes b' \# 1 \otimes a] \\ &\quad - \{x, a\} \otimes 1'_A \# 1 \otimes b + 1_A \otimes 1'_A \# 1 \otimes \{x, a\}b - 1_A \otimes b' \# 1 \otimes \{x, a\} \\ &\quad - a \otimes 1'_A \# 1 \otimes \{x, b\} + 1_A \otimes 1'_A \# 1 \otimes a\{x, b\} - 1_A \otimes \{x, b\}' \# 1 \otimes a \\ &= (a \otimes 1'_A \# x) \otimes b - (1_A \otimes 1'_A \# x) \otimes ab + (1_A \otimes b' \# x) \otimes a + (1_A \otimes 1'_A \# 1) \otimes \{x, a\}b \\ &\quad - (1_A \otimes b' \# 1) \otimes \{x, a\} + (1_A \otimes 1'_A \# 1) \otimes a\{x, b\} - (1_A \otimes \{x, b\}' \# 1) \otimes a \\ &= -\delta_{0,1}^H((a \otimes 1'_A \# 1) \otimes b \wedge x - (1_A \otimes 1'_A \# 1) \otimes ab \wedge x + (1_A \otimes b' \# 1) \otimes a \wedge x) \\ &= -(\delta_{0,1}^H \delta_{0,2}^V)(1_A \otimes 1'_A \# 1 \otimes a \otimes b \otimes x), \end{aligned}$$

which completes the proof. \square

3.2. The Poisson cohomology. Let M be a Poisson module over A . Applying the functor $\text{Hom}_{\mathcal{P}}(-, M)$ to the bicomplex $C_{\bullet,\bullet}$ we obtain a bicomplex of \mathbb{K} -vector spaces, which is called the *Poisson bicomplex for A with coefficients in M* and denoted by $PC^{\bullet,\bullet}(A; M)$. Clearly, the total complex $\text{Tot}(PC^{\bullet,\bullet}(A; M))$ is isomorphic to $\text{Hom}_{\mathcal{P}}(\chi_{\bullet}(A), M)$.

Following [3], we have a bicomplex $\tilde{C}^{\bullet,\bullet}(A; M)$ which is of the form

$$\begin{array}{ccccccc}
 & \dots & & \dots & & \dots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & \text{Hom}(A^3, M) & \xrightarrow{\delta_H} & \text{Hom}(A^3 \otimes \wedge^1, M) & \xrightarrow{\delta_H} & \text{Hom}(A^3 \otimes \wedge^2, M) \longrightarrow \dots \\
 & \delta_V \uparrow & & \delta_V \uparrow & & \uparrow \delta_V & \\
 0 & \longrightarrow & \text{Hom}(A^2, M) & \xrightarrow{\delta_H} & \text{Hom}(A^2 \otimes \wedge^1, M) & \xrightarrow{\delta_H} & \text{Hom}(A^2 \otimes \wedge^2, M) \longrightarrow \dots, \\
 & \delta_v \uparrow & & \delta_v \uparrow & & \uparrow \delta_v & \\
 M & \xrightarrow{\delta_h} & \text{Hom}(\wedge^1, M) & \xrightarrow{\delta_h} & \text{Hom}(\wedge^2, M) & \xrightarrow{\delta_h} & \text{Hom}(\wedge^3, M) \longrightarrow \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

where δ_h is the Chevalley-Eilenberg coboundary, δ_V is the Hochschild coboundary, δ_v is the composition of the natural homomorphism $\text{Hom}(\wedge^j, M) \hookrightarrow \text{Hom}(A \otimes \wedge^{j-1}, M)$ and the Hochschild coboundary $\text{Hom}(A \otimes \wedge^{j-1}, M) \rightarrow \text{Hom}(A^2 \otimes \wedge^{j-1}, M)$, and $\delta_H : \text{Hom}(A^i \otimes \wedge^{j-1}, M) \rightarrow \text{Hom}(A^i \otimes \wedge^j, M)$ is given by

$$\begin{aligned}
 & (\delta_H f)(a_1 \otimes \dots \otimes a_i \otimes (x_1 \wedge \dots \wedge x_j)) \\
 &= \sum_{l=1}^j (-1)^{l+1} \left(\{x_l, f(a_1 \otimes \dots \otimes a_i \otimes (x_1 \wedge \dots \wedge \hat{x}_l \dots \wedge x_j))\}_* \right. \\
 & \quad \left. - \sum_{t=1}^i f(a_1 \otimes \dots \otimes \{x_l, a_t\} \otimes \dots \otimes a_i \otimes (x_1 \wedge \dots \wedge \hat{x}_l \dots \wedge x_j)) \right) \\
 & \quad + \sum_{1 \leq p < q \leq j} (-1)^{p+q} f(a_1 \otimes \dots \otimes a_i \otimes (\{x_p, x_q\} \wedge x_1 \wedge \dots \wedge \hat{x}_p \dots \wedge \hat{x}_q \dots \wedge x_j)).
 \end{aligned}$$

Remark 3.2. Note that the coboundary δ_H is essentially the Chevalley-Eilenberg coboundary. In fact, $\text{Hom}(A^i \otimes \wedge^j, M) \cong \text{Hom}(\wedge^j, A^i \otimes M)$ and $A^i \otimes M$ is a Lie module over A with the action given by

$$\begin{aligned}
 \{x, a_1 \otimes \dots \otimes a_i \otimes m\}_* &= \sum_{t=1}^i a_1 \otimes \dots \otimes \{x, a_t\} \otimes \dots \otimes a_i \otimes m \\
 & \quad + a_1 \otimes \dots \otimes a_i \otimes \{x, m\}_*
 \end{aligned}$$

for any $x \in A$ and any $a_1 \otimes \dots \otimes a_i \otimes m \in A^i \otimes M$.

Moreover, by identifying $\text{Hom}_{\mathcal{P}}(\mathcal{P} \otimes A^i \otimes \wedge^j, M)$ with $\text{Hom}(A^i \otimes \wedge^j, M)$, we easily deduce the following result.

Proposition 3.3. *The double complex $PC^{\bullet,\bullet}(A; M)$ is isomorphic to $\tilde{C}^{\bullet,\bullet}(A; M)$.*

The total complex $\text{Hom}_{\mathcal{P}}(\chi_{\bullet}(A), M)$ thus has the form

$$\begin{aligned}
 0 \rightarrow M & \xrightarrow{d^0} \text{Hom}(A, M) \xrightarrow{d^1} \text{Hom}(A^2 \oplus \wedge^2, M) \xrightarrow{d^2} \text{Hom}\left(\bigoplus_{\substack{i+j=3 \\ i \neq 1}} A^i \otimes \wedge^j, A\right) \\
 & \rightarrow \dots \rightarrow \text{Hom}\left(\bigoplus_{\substack{i+j=n \\ i \neq 1}} A^i \otimes \wedge^j, M\right) \xrightarrow{d^n} \text{Hom}\left(\bigoplus_{\substack{i+j=n+1 \\ i \neq 1}} A^i \otimes \wedge^j, M\right) \rightarrow \dots,
 \end{aligned}$$

and we call it the *Poisson complex with coefficients in M* . Its n -th cohomology group is called the n -th *FGV-Poisson cohomology of A with coefficients in M* and denoted by $\text{HP}^n(A; M)$. The Poisson complex with coefficients in A is called the *Poisson complex of A* , and we simply denote $\text{HP}^n(A; A)$ by $\text{HP}^n(A)$ and call it the n -th *FGV-Poisson cohomology group of A* .

Remark 3.4. Clearly, $\text{HP}^0(A; M) = \{a \in A \mid \{a, m\}_* = 0, \forall m \in M\}$. In particular, $\text{HP}^0(A)$ is the center of the Lie algebra A .

$\text{HP}^1(A; M) = \text{PDer}(A; M)/\text{IPDer}(A; M)$ is the outer Poisson derivations of A with coefficients in M , where $\text{PDer}(A; M)$ is the set of Poisson derivations and $\text{IPDer}(A; M) = \{\{-, m\}_* \mid m \in M\}$ the set of inner ones. Recall that a linear map from A to M is called a Poisson derivation if it is simultaneously a derivation in the Lie algebra sense and in the associative sense.

For any $f = (f_1, f_0) \in \text{Ker}d^2$, we may define a new Poisson algebra $A \ltimes_f M$, which is called the *extension of A by M along f* . As a \mathbb{K} -vector space, $A \ltimes_f M = A \oplus M$; and the associative multiplication and the Lie bracket are given by

$$\begin{aligned} (a, x) \cdot (a', x') &= (aa', ax' + xa' + f_1(a \otimes a')), \\ \{(a, x), (a', x')\} &= (\{a, a'\}, \{a, x'\}_* - \{a', x\}_* + f_0(a \wedge a')). \end{aligned}$$

We have the following standard result, compare also with the deformation theory.

Proposition 3.5. *For any $f \in \text{Ker}d^2$, $A \ltimes_f M$ is a Poisson algebra. Moreover, for any $f, g \in \text{Ker}d^2$ with $\bar{f} = \bar{g}$ in $\text{HP}^2(A, M)$, there is an isomorphism of Poisson algebras $A \ltimes_f M \simeq A \ltimes_g M$.*

4. FORMAL DEFORMATIONS OF POISSON ALGEBRAS

In [3, Section 6], a deformation theory for Leibniz pairs was introduced. As a special case of Leibniz pairs, formal deformations for a Poisson algebra were also mentioned there with less detail. We mention that for a Poisson algebra, the deformations as a Poisson algebra are different from the ones as a Leibniz pair.

On the other hand, even for commutative Poisson algebras, the noncommutative version of deformation theory could be quite useful. For instance, Kontsevich's deformation quantization is explained as a special case of formal deformations in the noncommutative sense, and by using Poisson cohomology group we can give some necessary condition for the existence of deformation quantizations. In these reasons, we would give more details on Poisson algebra deformations here.

4.1. Formal Poisson deformations. Let $(A, \cdot, \{-, -\})$ be a Poisson algebra over \mathbb{K} . For consistency of notations the associative multiplication is denoted by m_0 and the Lie bracket by l_0 . Let $\mathbb{K}[[t]]$ be the formal power series ring in one variable t , and let $A[[t]] = A \otimes \mathbb{K}[[t]]$ be space of formal power series with coefficients in A . Clearly A embeds into $A[[t]]$ in a natural way, and any \mathbb{K} -bilinear map $A \times A \rightarrow A$ extends to a $\mathbb{K}[[t]]$ -bilinear map from $A[[t]] \times A[[t]]$ to $A[[t]]$. Via the natural isomorphism $A[[t]]/(tA[[t]]) \cong A$, A is also viewed as a quotient space of $A[[t]]$.

By a *formal deformation* of the Poisson algebra A it is meant a Poisson $\mathbb{K}[[t]]$ -algebra structure on $A[[t]]$, such that A is a quotient of $A[[t]]$ as Poisson algebras. Note that $tA[[t]]$ is a Poisson ideal and hence $A[[t]]/tA[[t]]$ inherits a Poisson algebra structure.

Since all structure maps are $\mathbb{K}[[t]]$ -linear, they are uniquely determined by the restriction to the subspace A . To be precise, let $m_t: A[[t]] \times A[[t]] \rightarrow A[[t]]$ and $l_t: A[[t]] \times A[[t]] \rightarrow A[[t]]$ be the associative multiplication and the Lie bracket respectively. Clearly, when restricted to $A \times A$, m_t and l_t can be rewritten as

$$\begin{aligned} m_t &= m_0 + tm_1 + t^2m_2 + \cdots, \\ l_t &= l_0 + tl_1 + t^2l_2 + \cdots, \end{aligned}$$

where $m_i, l_i: A \times A \rightarrow A$ are \mathbb{K} -bilinear maps for each i . The isomorphism $A[[t]]/tA[[t]] \cong A$ as Poisson algebras just says that the degree 0 parts in the above expansions are exactly given by the structure maps on A .

By definition, m_t and l_t induce formal deformations of A as an associative algebra and as a Lie algebra respectively. Clearly l_t is skew-symmetric, which is equivalent to all l_i 's are skew symmetric. Furthermore, the associativity, the Leibniz rule and the Jacobi identity read as

$$\begin{aligned} m_t(m_t(a, b), c) &= m_t(a, m_t(b, c)), \\ l_t(m_t(a, b), c) &= m_t(a, l_t(b, c)) + m_t(l_t(a, c), b), \\ l_t(l_t(a, b), c) + l_t(l_t(b, c), a) + l_t(l_t(c, a), b) &= 0, \end{aligned}$$

for all $a, b, c \in A$, which is rewritten as

$$(D_n-1) \quad \sum_{\substack{p+q=n \\ p, q > 0}} (m_p(m_q(a, b), c) - m_p(a, m_q(b, c)))$$

$$= am_n(b, c) - m_n(ab, c) + m_n(a, bc) - m_n(a, b)c,$$

$$(D_n-2) \quad \sum_{\substack{p+q=n \\ p, q > 0}} (l_q(m_p(a, b), c) - m_p(a, l_q(b, c)) - m_p(l_q(a, c), b))$$

$$\begin{aligned} &= al_n(b, c) - l_n(ab, c) + l_n(a, c)b \\ &\quad + \{c, m_n(a, b)\} - m_n(a, \{c, a\}) - m_n(\{c, a\}, b), \end{aligned}$$

$$(D_n-3) \quad \sum_{\substack{p+q=n \\ p, q > 0}} [l_q(l_p(a, b), c) + l_q(l_p(b, c), a) + l_q(l_p(c, a), b)]$$

$$\begin{aligned} &= l_n(a, \{b, c\}) + l_n(b, \{c, a\}) + l_n(c, \{a, b\}) \\ &\quad - \{l_n(a, b), c\} - \{l_n(b, c), a\} - \{l_n(c, a), b\} \end{aligned}$$

for all $n \geq 1$. (D_n-1), (D_n-2) and (D_n-3) are called the *deformation equations* for Poisson algebras. By definition $m_0(a, b) = ab$ and $l_0(a, b) = \{a, b\}$. The pair (m_1, l_1) is called an *infinitesimal deformation* of A , which gives a Poisson structure on the quotient space $A[[t]]/(t^2)$.

We assume that

$$\begin{aligned} F_1(a, b, c) &= \sum_{\substack{p+q=n \\ p, q > 0}} (m_p(m_q(a, b), c) - m_p(a, m_q(b, c))), \\ F_2(a, b, c) &= \sum_{\substack{p+q=n \\ p, q > 0}} (l_q(m_p(a, b), c) - m_p(a, l_q(b, c)) - m_p(l_q(a, c), b)), \\ F_3(a, b, c) &= \sum_{\substack{p+q=n \\ p, q > 0}} (l_q(l_p(a, b), c) + l_q(l_p(b, c), a) + l_q(l_p(c, a), b)). \end{aligned}$$

Similar to the associative algebra case, deformations of Poisson algebras will meet some *obstructions* in Poisson cohomology, which are all deducible from the deformation equations. In summary, we have the following result which essentially goes to Flato, Gerstenhaber and Voronov, although it was not written down explicitly.

Theorem 4.1. [3, Section 6] *Let $(A, \cdot, \{-, -\})$ be a Poisson algebra and $HP^n(A)$ the n -th FGV-Poisson cohomology group of A .*

(i) *The pair (m_1, l_1) is an infinitesimal deformation of A if and only if (m_1, l_1) is a 2-cocycle in the Poisson complex of A .*

(ii) If (m_i, l_i) satisfies (D_i-1) , (D_i-2) and (D_i-3) for $i = 1, \dots, n-1$, then (F_1, F_2, F_3) is a 3-cocycle, and hence $\text{HP}^3(A)$ is the $(n-1)$ -th “obstruction cocycle”. In particular, if $\text{HP}^3(A) = 0$, then all obstructions vanish.

Proof. (i) The pair (m_1, l_1) is an infinitesimal deformation of A if and only if it satisfies the deformation equations of case $n = 1$, i.e.

$$\begin{aligned} am_1(b, c) - m_1(ab, c) + m_1(a, bc) - m_1(a, b)c &= 0, \\ al_1(b, c) - l_1(ab, c) + l_1(a, c)b + \{c, m_1(a, b)\} - m_1(a, \{c, a\}) - m_1(\{c, a\}, b) &= 0, \\ l_1(a, \{b, c\}) + l_1(b, \{c, a\}) + l_1(c, \{a, b\}) - \{l_1(a, b), c\} - \{l_1(b, c), a\} - \{l_1(c, a), b\} &= 0, \end{aligned}$$

which say that (m_1, l_1) is a 2-cocycle in Poisson complex of A .

(ii) The deformation equations can be rewritten in the form

$$\begin{aligned} (\text{D}_n-1') \quad \sum_{\substack{p+q=n \\ p, q > 0}} (m_p(m_q(a, b), c) - m_p(a, m_q(b, c))) &= d^2(m_n, l_n)_3(a, b, c), \\ (\text{D}_n-2') \quad \sum_{\substack{p+q=n \\ p, q > 0}} (l_q(m_p(a, b), c) - m_p(a, l_q(b, c)) - m_p(l_q(a, c), b)) &= d^2(m_n, l_n)_2(a, b, c), \\ (\text{D}_n-3') \quad \sum_{\substack{p+q=n \\ p, q > 0}} (l_q(l_p(a, b), c) + l_q(l_p(b, c), a) + l_q(l_p(c, a), b)) &= d^2(m_n, l_n)_0(a, b, c). \end{aligned}$$

We only prove the case of $n = 2$ and the general cases can be similarly shown. In this case,

$$\begin{aligned} F_1(a, b, c) &= m_1(m_1(a, b), c) - m_1(a, m_1(b, c)), \\ F_2(a, b, c) &= l_1(m_1(a, b), c) - m_1(a, l_1(b, c)) - m_1(l_1(a, c), b), \\ F_3(a, b, c) &= l_1(l_1(a, b), c) + l_1(l_1(b, c), a) + l_1(l_1(c, a), b). \end{aligned}$$

Viewing (F_1, F_2, F_3) as an element in $\text{Hom}(A^3 \oplus A^2 \otimes \wedge^1 \oplus \wedge^3, A)$, it follows from some routine calculation that (F_1, F_2, F_3) is a 3-cocycle since (m_1, l_1) is a 2-cocycle. By (D_2-1') , (D_2-2') and (D_2-3') , the cohomology class of this element is the “obstruction” to the existence of (m_2, l_2) . In particular, if $\text{HP}^3(A) = 0$, then the obstruction vanishes. \square

Remark 4.2. We emphasize that $\text{HP}^3(A) \neq 0$ does not deny the existence of formal Poisson deformations. See the last section for an example.

Remark 4.3. Two deformations A_t and A'_t of A are said to be *equivalent*, if there exists an isomorphism of Poisson $\mathbb{K}[[t]]$ -algebras $g_t: A'_t \rightarrow A_t$ with $g_t(a) \in a + tA[[t]]$ for any $a \in A$. A Poisson algebra is *rigid* if any deformation is equivalent to a trivial one, that is, $m_i = 0$ and $l_i = 0$ for all $i \geq 1$. If $\text{HP}^2(A) = 0$, then A is rigid.

4.2. Deformation quantization. In this part, we will show that Kontsevich’s deformation quantization on Poisson manifolds can be understood as a special case of formal deformation of Poisson algebras in our sense. We refer to [2, 6, 9] for more details about deformation quantization theory.

Let (A, \cdot) be a commutative associative algebra, and $(A[[t]], m_t)$ a formal deformation of A as an associative algebra. As usual, m_t is given by a family of \mathbb{K} -bilinear functions $\{m_i: A \times A \rightarrow A\}$ with

$$m_t(a, b) = m_0(a, b) + tm_1(a, b) + t^2m_2(a, b) + \dots$$

for any $a, b \in A$, where for consistency of notations, $m_0(a, b) = a \cdot b$. Define the bilinear map $\{-, -\}: A \times A \rightarrow A$ by setting $\{a, b\} = m_1(a, b) - m_1(b, a)$. It is easy to check that $P = (A, \cdot, \{-, -\})$ is a commutative Poisson algebra.

Definition 4.4. [2, Definition 8.4] Let A, P be as above. Then P is called the *classical limit* of the associative product m_t , and $(A[[t]], m_t)$ a *deformation quantization* of the Poisson algebra P .

We emphasize that in geometric situation, say if A is the algebra $C^\infty(M)$ of smooth functions over some Poisson manifold M , then to define a quantization deformation, each $m_i: A \times A \rightarrow A$ in the expression

$$m_t = m_0 + tm_1 + t^2m_2 + \cdots$$

is required to be a polydifferential operator, i.e., a derivator in each argument.

Since (A, \cdot) is a commutative associative algebra, we may consider the standard Poisson structure (but with a scalar) on $(A[[t]], m_t)$. In fact, we may define a bracket

$$l_t = \frac{[-, -]}{t}: A[[t]] \times A[[t]] \rightarrow A[[t]],$$

where $[-, -]$ is the commutator of m_t . Clearly, $(A[[t]], m_t, l_t)$ is a Poisson algebra, and gives a formal deformation of the Poisson algebra $(A, \cdot, \{-, -\})$. Thus a deformation quantization of a commutative Poisson algebra can be understood as a special kind of formal deformation, say, the Poisson bracket is given by the commutator of the associative product with a scalar $\frac{1}{t}$. The following observation shows us an immediate advantage to take this viewpoint.

Proposition 4.5. *Let $P = (A, \cdot, \{-, -\})$ be a nontrivial commutative Poisson algebra. If $\text{HP}^2(A) = 0$, then P has no deformation quantization.*

Proof. Assume that $(A[[t]], m_t)$ is a deformation quantization of P . Then $(A[[t]], m_t, l_t)$ is a formal deformation of $(A, \cdot, \{-, -\})$, where

$$l_t(a, b) = \frac{1}{t}(m_t(a, b) - m_t(b, a))$$

for all $a, b \in A$. Since $\text{HP}^2(A) = 0$, the formal deformation $(A[[t]], m_t, l_t)$ is equivalent to the trivial one, that is, there exists an isomorphism

$$g: (A[[t]], m_t, l_t) \rightarrow (A[[t]], m'_t, l'_t)$$

of Poisson $\mathbb{K}[[t]]$ -algebras such that $g(a) \in a + tA[[t]]$ for any $a \in A$, where $m'_t(a, b) = ab$, $l'_t(a, b) = \{a, b\}$ for $a, b \in A$. Therefore,

$$\begin{aligned} \{a, b\} &= l'_t(a, b) = g(l_t(g^{-1}(a), g^{-1}(b))) \\ &= g\left(\frac{1}{t}(m_t(g^{-1}(a), g^{-1}(b)) - m_t(g^{-1}(b), g^{-1}(a)))\right) \\ &= \frac{1}{t}(m'_t(a, b) - m'_t(b, a)) = \frac{1}{t}(ab - ba) = 0, \end{aligned}$$

which leads to a contradiction. \square

Remark 4.6. Naively, by modifying the scalar in the definition, say replacing $\frac{1}{t}$ with $\frac{1}{t^n}$ (or some other $f(t) \in \mathbb{K}((t))$ in general), it is easy to define certain deformation quantization theory of higher orders, although the significance of this notion is not known to us at moment.

4.3. More types of Formal deformations. So far, we have discussed the deformations in which the associative multiplication and the Lie bracket are deformed simultaneously. There are some other choices, say one can also fix one structure map and deform the another one. We would say some words on this direction. We mention that similar work for commutative Poisson algebras can be found in [4, 8, 9].

A formal deformation $(A[[t]], m_t, l_t)$ of a Poisson algebra A is said to be of *type I* (resp. *of type II*), if $m_i = 0$ (resp. $l_i = 0$) for all $i \geq 1$.

Let $(A[[t]], m_t, l_t)$ be a formal deformation of type I. For any $n \geq 1$, the deformation equation (D_n-1) holds automatically, and (D_n-2) just says that $l_n: A \times A \rightarrow A$ is a biderivation of A .

We simply denote the Poisson bicomplex $PC^{\bullet,\bullet}(A; A)$ by $PC^{\bullet,\bullet}$. Considering the spectral sequence induced by the first filtration of this bicomplex, we obtain a complex

$$0 \rightarrow A \rightarrow H_1^0(PC^{\bullet,1}) \rightarrow H_1^0(PC^{\bullet,2}) \rightarrow \cdots \rightarrow H_1^0(PC^{\bullet,j}) \rightarrow H_1^0(PC^{\bullet,j+1}) \rightarrow \cdots,$$

where

$$H_1^0(PC^{\bullet,j}) = \{f \in \text{Hom}(\wedge^j, A) \mid \delta_v(f) = 0\}$$

is the space of all skew-symmetric n -fold derivations of A . Clearly, the above complex is a noncommutative version of the Lichnerowicz-Poisson complex, and its cohomology groups, i.e. $E_{112}^{0,q} = H_{11}^q H_1^0(PC^{\bullet,\bullet})$ in the first spectral sequence induced by $PC^{\bullet,\bullet}$, control the deformations of type I.

Similarly, the second filtration of $PC^{\bullet,\bullet}$ gives a complex

$$0 \rightarrow H_{11}^1(PC^{1,\bullet}) \rightarrow H_{11}^1(PC^{2,\bullet}) \rightarrow \cdots \rightarrow H_{11}^1(PC^{i,\bullet}) \rightarrow H_{11}^1(PC^{i+1,\bullet}) \rightarrow \cdots,$$

where

$$H_{11}^1(PC^{i,\bullet}) = \{f \in \text{Hom}(A^i, A) \mid \delta_h(f) = 0\}.$$

and its cohomology groups, i.e. $E_{12}^{p,1} = H_1^p H_{11}^1(PC^{\bullet,\bullet})$ in the second spectral sequence, control the formal deformation of type II.

5. COMPARISON WITH LICHNEROWICZ-POISSON COHOMOLOGY

In this section, we assume that A is a commutative Poisson algebra. Recall the definition of Lichnerowicz-Poisson cohomology for a commutative Poisson algebra, see [4, 5, 7, 8, 9] for detail. We set $\chi^i(A) = 0$ for any $i < 0$, $\chi^0(A) = A$ and $\chi^1(A) = \text{Der}(A)$. For $i \geq 2$, let $\chi^\bullet(A) \subseteq \text{Hom}(\wedge^\bullet A, A)$ be the subset consisting of all skew-symmetric multiderivations of A . We may form a complex $(\chi^\bullet(A), \delta_{LP})$, where δ_{LP} is given by

$$\begin{aligned} & \delta_{LP}^n f(a_0 \wedge a_1 \wedge \cdots \wedge a_n) \\ &= \sum_{i=0}^n (-1)^i \{a_i, f(a_0 \wedge \cdots \widehat{a_i} \cdots \wedge a_n)\} \\ &+ \sum_{0 \leq p < q \leq n} (-1)^{p+q} f(\{a_p, a_q\} \wedge a_0 \wedge \cdots \widehat{a_p} \cdots \widehat{a_q} \cdots \wedge a_n). \end{aligned}$$

This complex is called the *Lichnerowicz-Poisson complex* of A , or simply *LP-complex*, and its n -th cohomology group, denoted by $H_{LP}^n(A)$, is called the *n-th Lichnerowicz-Poisson cohomology group* of A , see [8].

Proposition 5.1. *Let A be a commutative Poisson algebra. Then the family of \mathbb{K} -linear functions $\{\sigma^n, n \in \mathbb{N}\}$,*

$$\sigma^n : \chi^n(A) \rightarrow \text{Hom}\left(\bigoplus_{\substack{i+j=n \\ i \neq 1}} A^i \otimes \wedge^j, A\right)$$

$$\sigma^n(f) = (0, \dots, 0, f)$$

is a chain map from the Lichnerowicz-Poisson complex to the Poisson complex.

Proof. Clearly,

$$d^n \sigma^n f = (0, \dots, 0, \delta_{CE} f) = \sigma^{n+1} \delta_{LP} f,$$

where δ_{CE} is the Chevalley-Eilenberg coboundary. Therefore, $\{\sigma^n, n \in \mathbb{N}\}$ is a chain map from LP-complex to Poisson complex. \square

Remark 5.2. Easy computation shows that the 0-th and 1-th Lichnerowicz-Poisson cohomology groups are the same as the ones of Poisson cohomology, respectively.

Moreover, by direct calculation, we show that the Lichnerowicz-Poisson cohomology relates to the FGV-Poisson cohomology closely.

Theorem 5.3. *Let $(A, \cdot, \{-, -\})$ be a commutative Poisson algebra. Then*

$$H_{LP}^n(A) \cong E_{II2}^{0,n},$$

where $E_{II2}^{0,n}$ is the $(n, 0)$ -term in the spectral sequence induced by the first filtration of the Poisson bicomplex.

Remark 5.4. In some extreme case, say if A has trivial Lie bracket, then the Poisson cohomology and the Lichnerowicz-Poisson cohomology is known, and hence the difference between them is very clear. In fact, in this case

$$\begin{aligned} H_{LP}^n(A) &= \chi^n(A), \\ HP^n(A) &= \bigoplus_{i=2}^n (HH^i(A) \otimes \wedge^{n-i}) \bigoplus \chi^n(A). \end{aligned}$$

6. A CHARACTERIZATION OF POISSON COHOMOLOGY GROUPS

In section 3, we have introduced the characteristic complex $\chi_\bullet(A)$ of the Poisson algebra A and used it study the Poisson cohomology. However, whether $\chi_\bullet(A)$ is quasi-isomorphic to a Poisson module, is not known to us yet. We will make some discussion on this question.

Let \mathcal{Q} be the quasi-Poisson enveloping algebra of $(A, \cdot, \{-, -\})$ and $\Omega^2(A)$ be the second syzygy of A as an A^e -module. More precisely, $\Omega^2(A)$ is the quotient module of A^e -module A^4 modulo the submodule generated by $\{a \otimes b \otimes c \otimes 1_A - 1_A \otimes ab \otimes c \otimes 1_A + 1 \otimes a \otimes bc \otimes 1_A - 1 \otimes a \otimes b \otimes c \mid a, b, c \in A\}$. Note that $\Omega^2(A)$ is also a quotient module of A^4 as a \mathcal{Q} -module. Similar to the construction of the free \mathcal{Q} -resolution of A [1], we consider the standard resolution

$$\mathcal{S}_\bullet : \dots \rightarrow A^{i+4} \rightarrow A^{i+3} \rightarrow \dots \rightarrow A^5 \rightarrow A^4 \rightarrow \Omega^2(A) \rightarrow 0$$

of $\Omega^2(A)$, and the Koszul resolution of \mathbb{K} as a $\mathcal{U}(A)$ -module

$$\mathcal{K}_\bullet : \dots \rightarrow \mathcal{U}(A) \otimes \wedge^j \rightarrow \mathcal{U}(A) \otimes \wedge^{j-1} \rightarrow \dots \rightarrow \mathcal{U}(A) \otimes \wedge^1 \rightarrow \mathcal{U}(A) \rightarrow \mathbb{K} \rightarrow 0.$$

Denote by $\widehat{\mathcal{S}}_\bullet$ and $\widehat{\mathcal{K}}_\bullet$ the corresponding deleted resolutions.

By taking the total complex of the tensor product of $\widehat{\mathcal{S}}_\bullet$ and $\widehat{\mathcal{K}}_\bullet$, we obtain a free \mathcal{Q} -resolution of $\Omega^2(A)$

$$\begin{aligned} \mathcal{T}'_\bullet : \dots \rightarrow \bigoplus_{i+j=n} A^{i+4} \otimes \mathcal{U}(A) \otimes \wedge^j &\rightarrow \bigoplus_{i+j=n-1} A^{i+4} \otimes \mathcal{U}(A) \otimes \wedge^j \rightarrow \dots \\ &\rightarrow A^5 \otimes \mathcal{U}(A) \oplus A^4 \otimes \mathcal{U}(A) \otimes \wedge^1 \rightarrow A^4 \otimes \mathcal{U}(A) \rightarrow \Omega^2(A) \rightarrow 0. \end{aligned}$$

On the other hand, we may apply the functor $A^2 \otimes -$ to the deleted resolution $\widehat{\mathcal{K}}_\bullet$, and obtain a free resolution of \mathcal{Q} -module A^2

$$\mathcal{T}''_\bullet : \dots \rightarrow A^2 \otimes \mathcal{U}(A) \otimes \wedge^n \rightarrow A^2 \otimes \mathcal{U}(A) \otimes \wedge^{n-1} \rightarrow \dots \rightarrow A^2 \otimes \mathcal{U}(A) \rightarrow A^2 \rightarrow 0.$$

By definition of $C_{\bullet, \bullet}$, we have a short exact sequence of complexes

$$0 \rightarrow \mathcal{P} \otimes_{\mathcal{Q}} \mathcal{T}_{\bullet}'' \rightarrow \chi_{\bullet}(A) \rightarrow \mathcal{P} \otimes_{\mathcal{Q}} \mathcal{T}_{\bullet}' \rightarrow 0,$$

thus we have the following long exact sequence, relating the homologies of the characteristic complex with some torsion groups of quasi-Poisson modules.

Proposition 6.1. *Keep the above notations. Then there is a long exact sequence*

$$\begin{aligned} \cdots \rightarrow \mathrm{Tor}_n^{\mathcal{Q}}(\mathcal{P}, A^2) \rightarrow H_n(\chi_{\bullet}(A)) \rightarrow \mathrm{Tor}_{n-2}^{\mathcal{Q}}(\mathcal{P}, \Omega^2(A)) \rightarrow \mathrm{Tor}_{n-1}^{\mathcal{Q}}(\mathcal{P}, A^2) \rightarrow \cdots \\ \rightarrow \mathrm{Tor}_1^{\mathcal{Q}}(\mathcal{P}, A^2) \rightarrow H_1(\chi_{\bullet}(A)) \rightarrow 0 \rightarrow \mathcal{P} \otimes A^2 \rightarrow H_0(\chi_{\bullet}(A)) \rightarrow 0. \end{aligned}$$

Now we move to the calculation of Poisson cohomology groups. Firstly, we deduce from the free \mathcal{Q} -resolutions \mathcal{T}_{\bullet}' and \mathcal{T}_{\bullet}'' the following results.

Lemma 6.2. *Let M be a quasi-Poisson module over A . Then*

$$\mathrm{Ext}_{\mathcal{Q}}^n(\Omega^2(A), M) \cong H^n(QC^{\bullet}(\Omega^2(A), M)), \forall n \geq 0,$$

where $QC^{\bullet}(\Omega^2(A), M)$ is the complex

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}(A^2, M) \rightarrow \mathrm{Hom}(A^3 \oplus A^2 \otimes \wedge^1, M) \rightarrow \cdots \rightarrow \mathrm{Hom}\left(\bigoplus_{i+j=n-1} A^{i+2} \otimes \wedge^j, M\right) \\ \rightarrow \mathrm{Hom}\left(\bigoplus_{i+j=n} A^{i+2} \otimes \wedge^j, M\right) \rightarrow \cdots. \end{aligned}$$

Lemma 6.3. *Let M be a quasi-Poisson module. Then*

$$\mathrm{Ext}_{\mathcal{Q}}^n(A^2, M) \cong \mathrm{Ext}_{\mathcal{U}(A)}^n(\mathbb{K}, M).$$

Proof. Here we use the fact that A^2 is a free A^e -module. In fact, applying the isomorphisms $\mathrm{Hom}_{\mathcal{Q}}(A^2 \otimes_{\mathcal{U}(A)} \wedge^n, M) \cong \mathrm{Hom}(\wedge^n, M)$, we know that $\mathrm{Hom}_{\mathcal{Q}}(\mathcal{T}_{\bullet}'', M)$ is isomorphic to the Chevalley-Eilenberg complex

$$CE^{\bullet} : 0 \rightarrow M \rightarrow \mathrm{Hom}(\wedge^1, M) \rightarrow \mathrm{Hom}(\wedge^2, M) \rightarrow \cdots \rightarrow \mathrm{Hom}(\wedge^n, M) \rightarrow \cdots,$$

and the conclusion follows. \square

Denote $PC^{\bullet}(A, M)$ be the Poisson complex of A with coefficients in M . Clearly we have a short exact sequence of complexes

$$0 \rightarrow QC^{\bullet}(\Omega^2(A), M)[-2] \rightarrow PC^{\bullet}(A, M) \rightarrow CE^{\bullet} \rightarrow 0,$$

and combined with Lemma 6.2 and Lemma 6.3, we obtain the following long exact sequence.

Theorem 6.4. *Let A be a Poisson algebra and M a Poisson module over A . Then we have a long exact sequence*

$$\begin{aligned} (6.1) \quad 0 \rightarrow \mathrm{HP}^0(A, M) \rightarrow \mathrm{Hom}_{\mathcal{U}(A)}(\mathbb{K}, M) \rightarrow 0 \rightarrow \mathrm{HP}^1(A, M) \rightarrow \mathrm{Ext}_{\mathcal{U}(A)}^1(\mathbb{K}, M) \rightarrow \\ \mathrm{Hom}_{\mathcal{Q}}(\Omega^2(A), M) \rightarrow \mathrm{HP}^2(A, M) \rightarrow \mathrm{Ext}_{\mathcal{U}(A)}^2(\mathbb{K}, M) \rightarrow \cdots \rightarrow \mathrm{Ext}_{\mathcal{Q}}^{n-2}(\Omega^2(A), M) \\ \rightarrow \mathrm{HP}^n(A, M) \rightarrow \mathrm{Ext}_{\mathcal{U}(A)}^n(\mathbb{K}, M) \rightarrow \mathrm{Ext}_{\mathcal{Q}}^{n-1}(\Omega^2(A), M) \rightarrow \cdots. \end{aligned}$$

We recall a useful spectral sequence introduced in our previous work [1].

Proposition 6.5 ([1], Section 5). *Let M, N be modules over \mathcal{Q} . Then we have a spectral sequence*

$$\mathrm{Ext}_{\mathcal{U}(A)}^q(\mathbb{K}, \mathrm{Ext}_{A^e}^p(M, N)) \implies \mathrm{Ext}_{\mathcal{Q}}^{p+q}(M, N).$$

In conclusion, the above results provide us a way to read the information of Poisson cohomology from the Lie algebra cohomology and the Hochschild cohomology.

Example 6.6. Let A be the \mathbb{K} -algebra of upper triangular 2×2 matrices. This algebra is known to be the path algebra of the quiver of \mathbb{A}_2 type.

Consider the standard Poisson algebra. Clearly, A is a hereditary algebra as an associative algebra and hence $\mathrm{HH}^n(A) = 0$ for all $n \geq 1$. By direct computation, we have $\mathrm{HL}^0(A) = \mathbb{K}$, $\mathrm{HL}^1(A) = \mathbb{K}^2$, $\mathrm{HL}^2(A) = \mathbb{K}$, and $\mathrm{HL}^n(A) = 0$ for all $n \geq 3$, where $\mathrm{HL}^i(A) = \mathrm{Ext}_{\mathcal{U}(A)}^i(\mathbb{K}, A)$ for each i .

Applying the above spectral sequence and by some direct calculations, we know that $\mathrm{Hom}_{\mathcal{Q}}(\Omega^2(A), A) = \mathbb{K}^3$, $\mathrm{Ext}_{\mathcal{Q}}(\Omega^2(A), A) = \mathbb{K}^6$, $\mathrm{Ext}_{\mathcal{Q}}^2(\Omega^2(A), A) = \mathbb{K}^3$ and $\mathrm{Ext}_{\mathcal{Q}}^n(\Omega^2(A), A) = 0$ for all $n \geq 3$. Now it follows from Theorem 6.4 that

$$\mathrm{HP}^0(A) = \mathbb{K}, \mathrm{HP}^1(A) = 0, \mathrm{HP}^2(A) = \mathbb{K}, \mathrm{HP}^3(A) = \mathbb{K}^5, \mathrm{HP}^4(A) = \mathbb{K}^3,$$

and $\mathrm{HP}^n(A) = 0$ for all $n \geq 5$.

7. A FURTHER EXAMPLE: $\mathbb{M}_2(\mathbb{K})$

Let $A = \mathbb{M}_2(\mathbb{K})$ be the standard Poisson algebra, where $\mathbb{M}_2(\mathbb{K})$ is the algebra of 2×2 matrices with entries in \mathbb{K} . Clearly, as a Lie algebra, $A = \mathbb{K} \cdot 1_A \oplus \mathfrak{sl}_2(\mathbb{K})$, where $\mathbb{K} \cdot 1_A$ is an abelian Lie algebra of dimension 1 and $\mathfrak{sl}_2(\mathbb{K})$ is the special linear Lie algebra with the standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the module $\mathfrak{sl}_2(\mathbb{K}) \otimes \mathfrak{sl}_2(\mathbb{K})$ over the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$. It is easy to show the following decomposition as Lie modules:

$$\mathfrak{sl}_2(\mathbb{K}) \otimes \mathfrak{sl}_2(\mathbb{K}) = V_5 \oplus V_3 \oplus V_1,$$

where V_5, V_3 and V_1 are simple modules of dimension 5, 3 and 1 respectively. More precisely, V_5 has a \mathbb{K} -basis

$$\{e \otimes e, h \otimes e + e \otimes h, e \otimes f - h \otimes h + f \otimes e, f \otimes f, h \otimes f + f \otimes h\},$$

V_3 has a \mathbb{K} -basis

$$\{h \otimes e - e \otimes h, e \otimes f - f \otimes e, h \otimes f - f \otimes h\},$$

and V_1 has a \mathbb{K} -basis $\{2e \otimes f + h \otimes h + 2f \otimes e\}$, where all basis elements are given by weight vectors.

By some direct computation, we have the following well-known fact which will be useful in our later calculation.

Lemma 7.1. $\mathrm{Hom}_{\mathcal{U}(\mathfrak{sl}_2(\mathbb{K}))}(\mathfrak{sl}_2(\mathbb{K}) \otimes \mathfrak{sl}_2(\mathbb{K}), A) \cong \mathbb{K}^2$. More precisely, for any $\varphi \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{sl}_2(\mathbb{K}))}(\mathfrak{sl}_2(\mathbb{K}) \otimes \mathfrak{sl}_2(\mathbb{K}), A)$, there exist unique $\lambda, \mu \in \mathbb{K}$, such that

$$\varphi|_{V_5} = 0,$$

$$\varphi(h \otimes e - e \otimes h) = \lambda e, \varphi(e \otimes f - f \otimes e) = \frac{\lambda}{2} h, \varphi(f \otimes h - h \otimes f) = \lambda f,$$

and

$$\varphi(2e \otimes f + h \otimes h + 2f \otimes e) = \mu 1_A.$$

In fact, the image of φ on the basis of $\mathfrak{sl}_2(\mathbb{K}) \otimes \mathfrak{sl}_2(\mathbb{K})$ is shown in the following table.

Table 1. The image of φ on the basis of $\mathfrak{sl}_2(\mathbb{K}) \otimes \mathfrak{sl}_2(\mathbb{K})$

$\varphi(- \otimes -)$	e	f	h
e	0	$\frac{\mu}{6}1_A + \frac{\lambda}{4}h$	$-\frac{\lambda}{2}e$
f	$\frac{\mu}{6}1_A - \frac{\lambda}{4}h$	0	$\frac{\lambda}{2}f$
h	$\frac{\lambda}{2}e$	$-\frac{\lambda}{2}f$	$\frac{\mu}{3}1_A$

To compute the FGV-Poisson cohomology of A , we need also the following facts.

Lemma 7.2. *Keep the above notations. Then*

- (1) $\text{Ext}_{\mathcal{U}(A)}^2(\mathbb{K}, A) = 0$,
- (2) $\text{Ext}_{\mathcal{U}(A)}^1(\mathbb{K}, A) \cong \mathbb{K}$,
- (3) $\text{Hom}_{\mathcal{Q}}(\Omega^2(A), A) \cong \mathbb{K}^2$,

Proof. The proofs of (1) and (2) are just some routine calculations and are omitted here. We only prove part (3), which needs some technical argument.

By definition, $\Omega^2(A) = A^4/I$, where I is the submodule of the \mathcal{Q} -module A^4 generated by

$$\{a \otimes b \otimes c \otimes 1_A - 1 \otimes ab \otimes c \otimes 1_A + 1 \otimes a \otimes bc \otimes 1_A - 1 \otimes a \otimes b \otimes c\}.$$

Therefore, we know that

$$\text{Hom}_{\mathcal{Q}}(\Omega^2(A), A) \cong \{f \in \text{Hom}_{\mathcal{U}(A)}(A^2, A) \mid f \text{ satisfies } (*)\},$$

where $(*)$ means the equation

$$af(b, c) - f(ab, c) + f(a, bc) - f(a, b)c = 0, \quad (*)$$

or equivalently, f is a 2-cocycle in the Hochschild complex.

Now let φ be in $\text{Hom}_{\mathcal{U}(A)}(A^2, A)$ and satisfy $(*)$. Then we have $\varphi(1_A, 1_A) \in Z(A)$ and hence $\varphi(1_A, 1_A) = \nu 1_A$ for some $\nu \in \mathbb{K}$, the reason is that φ is a Lie module homomorphism, that is

$$\{a, \varphi(1_A \otimes 1_A)\} = \varphi(\{a, 1_A \otimes 1_A\}) = 0$$

holds for all $a \in A$. By applying $(*)$, we therefore obtain that for all $x \in A$,

$$\varphi(x \otimes 1_A) = \varphi(1_A \otimes x) = \nu x.$$

On the other hand, each Lie module can be viewed as a module over $\mathfrak{sl}_2(\mathbb{K})$ and

$$\text{Hom}_{\mathcal{U}(A)}(A^2, A) = \text{Hom}_{\mathcal{U}(\mathfrak{sl}_2(\mathbb{K}))}(A^2, A).$$

Now φ is determined by ν and its restriction to $\mathfrak{sl}_2(\mathbb{K}) \otimes \mathfrak{sl}_2(\mathbb{K})$, and the latter one is uniquely given by some λ and μ as shown in Lemma 7.1. By Table 1 and the equation $(*)$, we show that $\mu = 3\lambda$. Therefore φ is determined by ν and λ as shown in the following table.

Table 2. The image of φ on the \mathbb{K} -basis of $A \otimes A$

$\varphi(- \otimes -)$	1_A	e	f	h
1_A	$\nu 1_A$	νe	νf	νh
e	νe	0	$\frac{\lambda}{2}1_A + \frac{\lambda}{4}h$	$-\frac{\lambda}{2}e$
f	νf	$\frac{\lambda}{2}1_A - \frac{\lambda}{4}h$	0	$\frac{\lambda}{2}f$
h	νh	$\frac{\lambda}{2}e$	$-\frac{\lambda}{2}f$	$\lambda 1_A$

Conversely, any $\nu, \lambda \in \mathbb{K}$ uniquely give to an element $\Phi_{\nu, \lambda} \in \text{Hom}_{\mathcal{U}(A)}(A^2, A)$ which satisfies $(*)$, and hence an element in $\text{Hom}_{\mathcal{Q}}(\Omega^2(A), A)$. The proof is completed. \square

As a vector space, $\mathbb{M}_2(\mathbb{K})$ is of 4 dimension, and hence there are many choices of associative multiplication on it, among which are two extreme cases. One is given by the matrix product, and the other one is the trivial one given by setting

$$e^2 = f^2 = h^2 = ef = fe = eh = he = fh = hf = 0.$$

An interesting observation is that these two are essentially the only cases to make the general linear Lie algebra $\mathbb{M}_2(\mathbb{K})$ a Poisson algebra.

Corollary 7.3. *Let $(\mathbb{M}_2(\mathbb{K}), \circ, [-, -])$ be a Poisson algebra, where $(\mathbb{M}_2(\mathbb{K}), [-, -])$ is the general linear Lie algebra. Then as an associative algebra, $(\mathbb{M}_2(\mathbb{K}), \circ)$ is either isomorphic to the total matrix algebra, or to the trivial one.*

Proof. For simplicity, we set $P = \mathbb{M}_2(\mathbb{K})$ and $m(a, b) = a \circ b$ for $a, b \in P$. The Leibniz rule implies that m is a homomorphism of Lie modules from P^2 to P . From the proof of (4) in Lemma 7.2 and Table 1, we know $4\mu = 3\lambda^2$ since $m(a, m(b, c)) = m(m(a, b), c)$. When $\lambda = 0$, the associative algebra $(\mathbb{M}_2(\mathbb{K}), \circ)$ is the trivial associative algebra, and when $\lambda \neq 0$, it is isomorphic to the total matrix algebra. \square

By applying the long exact sequence in Theorem 6.4 and Lemma 7.2, the Poisson cohomology groups of A of lower degrees is calculated as follows.

Proposition 7.4. *Keep the above notations. Then*

- (1) $\text{HP}^0(A) \cong \mathbb{K}$;
- (2) $\text{HP}^1(A) = 0$;
- (3) $\text{HP}^2(A) \cong \mathbb{K}$.

More precisely, $\overline{(\Phi_{0,2}, 0)}$ gives a basis of $\text{HP}^2(A)$. In fact, by construction as in Table 2, $\Phi_{0,2}$ is a Lie module homomorphism and satisfies the condition (*), which implies that $(\Phi_{0,2}, 0) \in \text{Hom}(A^2 \oplus \wedge^2, A)$ is a Poisson 2-cocycle of A , and its corresponding cohomology class $\overline{(\Phi_{0,2}, 0)}$ gives an element in $\text{HP}^2(A)$.

For given $s \in \mathbb{K}$, we may define a $\mathbb{K}[[t]]$ -Poisson algebra structure on $A[[t]]$ by setting $l_i = 0$ for all $i \geq 0$, and m_t to be given as in the following table.

Table 3. The image of m_t on the basis of $A \otimes A$

$m_t(-, -)$	1_A	e	f	h
1_A	1_A	e	f	h
e	e	0	$\frac{1}{2}(1-t)h + \frac{1}{2}(1-ts)^2 1_A$	$-e - tse$
f	f	$-\frac{1}{2}(1-t)h + \frac{1}{2}(1-ts)^2 1_A$	0	$f + tsf$
h	h	$e - tse$	$-f - tsf$	$(1-ts)^2$

It is direct to show that $(A[[t]], m_t, l_t)$ is a formal Poisson deformation of A “lifting” the Poisson 2-cocycle $s(\Phi_{0,2}, 0) = (\Phi_{0,2s}, 0)$, say $(m_1, l_1) = (\Phi_{0,2s}, 0)$. Moreover, for any Poisson 2-cocycle η , there exists some $s \in \mathbb{K}$ such that $\bar{\eta} = \overline{(\Phi_{0,2s}, 0)}$ in $\text{HP}^2(A)$, a standard argument shows the existence of a formal deformation $(A[[t]], m'_t, l'_t)$ lifting η which is equivalent to $(A[[t]], m_t, l_t)$.

Remark 7.5. It is worth mentioning that for the given Poisson 2-cocyle $(\Phi_{0,2s}, 0)$, we only give one formal deformation. It is not known yet whether all formal deformations lifting it are equivalent.

By the long exact sequence (6.1) in Theorem 6.4 and some direct calculations, we obtain that $\text{HP}^3(A) \neq 0$. Therefore, the above example also tells that it is still possible to have formal deformations even though “obstructions” exist.

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Yan-Hong Bao

School of Mathematical Sciences, Anhui University, Hefei, China, 230039

School of Mathematical Sciences, University of Sciences and Technology of China, Hefei,China, 230036

E-mail address: yhbao@ustc.edu.cn

Yu Ye

School of Mathematical Sciences, University of Sciences and Technology of China, Hefei,China, 230036

E-mail address: yeyu@ustc.edu.cn